COINVARIANTS FOR YANGIAN DOUBLES AND QUANTUM KNIZHNIK-ZAMOLODCHIKOV EQUATIONS

B. ENRIQUEZ AND G. FELDER

Abstract. We present a quantum version of the construction of the KZ system of equations as a flat connection on the spaces of coinvariants of representations of tensor products of Kac-Moody algebras. We consider here representations of a tensor product of Yangian doubles and compute the coinvariants of a deformation of the subalgebra generated by the regular functions of a rational curve with marked points. We observe that Drinfeld's quantum Casimir element can be viewed as a deformation of the zero-mode of the Sugawara tensor in the Yangian double. These ingredients serve to define a compatible system of difference equations, which we identify with the quantum KZ equations introduced by I. Frenkel and N. Reshetikhin.

Introduction. The Knizhnik-Zamolodchikov (KZ) system is a set of differential equations satisfied by correlation functions in Wess-Zumino-Witten theories ([12]). These equations can be interpreted as the equations satisfied by matrix elements of intertwining operators associated with representations of affine Kac-Moody algebras ([14]). They define a local system on the configuration space of n distinct marked points on the rational curve $\mathbb{C}P^1$. This local system also has another interpretation: to a complex curve with marked points, a system of weights of a semisimple Lie algebra and a positive integer is associated the vector space of conformal blocks. It is the space of coinvariants of a representation of a product of Kac-Moody algebras attached to the points of the curve, with respect to the subalgebra formed by the rational functions on the curve, regular outside the points. The conformal blocks form a vector bundle on the moduli space of curves with marked points. There exists a natural connection on this vector bundle, the KZB (for KZ-Bernard) connection. It is provided by the action of the Sugawara field. This field is a generating functional for elements of the enveloping algebra of the Kac-Moody algebra. One shows that certain combinations of these elements conjugate the regular algebras associated to nearby elements of the moduli space (see [2, 15]).

In [11], I. Frenkel and N. Reshetikhin introduced q-deformed analogues of the KZ equations. This qKZ system is a difference system obeyed by matrix elements of intertwining operators of quantum affine algebras. Later, elliptic analogues of this system were defined and studied ([9, 10]). If one wished to understand q-deformed versions of the KZB connection in higher genus, it would be important

Date: July 1997.

to understand how these equations could be derived from the coinvariants viewpoint. To make such a derivation explicit in the rational case is the main goal of this paper.

Let us now present our work. We consider a system of points $\mathbf{z} = (z_i)_{i=1,\dots,n}$ on $\mathbb{C}P^1$. We call $\mathcal{O}_i$ and $\mathcal{K}_i$ the completed ring and field of $\mathbb{C}P^1$ at z_i. We also call $\mathcal{O}$ and $\mathcal{K}$ their direct sums, and $R_{\mathbf{z}}$ the ring of regular functions on $\mathbb{C}P^1$, regular outside $\mathbf{z}$ and vanishing at infinity. We set $\bar{\mathfrak{g}} = \mathfrak{sl}_2$, and we denote by $\mathfrak{g}_{\mathcal{K}}$ the double extension of $\bar{\mathfrak{g}} \otimes \mathcal{K}$ by central and derivation elements, and by $\mathfrak{g}_{\mathcal{O}}$ and $\mathfrak{g}_{\mathcal{K}}$ extensions $\bar{\mathfrak{g}} \otimes \mathcal{O}$ and of $\bar{\mathfrak{g}} \otimes R_{\mathbf{z}}$. In this situation, the coinvariants construction described above is based on the inclusions of the enveloping algebras $U\mathfrak{g}_{\mathcal{O}}$ and $U\mathfrak{g}_{\mathbf{z}}$ in $U\mathfrak{g}_{\mathcal{K}}$.

To construct a deformation of these inclusions, we note that the decomposition of $\mathfrak{g}_{\mathcal{K}}$ as a direct sum $\mathfrak{g}_{\mathcal{O}} \oplus \mathfrak{g}_{\mathbf{z}}$ is that of a Manin triple, associated with the rational form dz on $\mathbb{C}P^1$. We apply techniques of quantum currents and twists [7, 5] to the quantization of this triple (sect. 1.1). We then give an presentation of the resulting algebras $U_\hbar\mathfrak{g}_{\mathcal{O}}, U_\hbar\mathfrak{g}_{\mathbf{z}}$ and $U_\hbar\mathfrak{g}_{\mathcal{K}}$ in terms of L-operators (sect. 1.4). Let us mention here that P. Etingof and D. Kazhdan obtained in [8] quantizations of these triples, for arbitrary $\bar{\mathfrak{g}}$; their construction of $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ from the double Yangian algebra is a special case of their construction of "factored algebras", that applies to any quantum group based on an 1-dimensional algebraic group. They also obtain RTT-type relations similar to our RLL relations ([8], Prop. 3.25).

Our next step is to construct an isomorphism between $U_\hbar\mathfrak{g}_{\mathcal{K}}$ and a tensor product of n copies of the double Yangian algebra $DY(\mathfrak{sl}_2)$, with their central elements identified. This is done in Prop. 1.6 using the L-operators description of both algebras. The formulas closely resemble formulas for coproducts. They may have the following interpretation: the isomorphism of the quantum groups appearing in [6] with a tensor product of "local" algebras could be obtained if we extended the construction of that paper to a larger extension, with a n-dimensional center and n derivations. The resulting algebra would then have specialization morphisms to local algebras, and the desired isomorphism would result from composing the coproduct $\Delta^{(n)}$ with the tensor product of these specialization morphisms. We hope to return to this question elsewhere.

After that, we observe that there exists in the double Yangian $DY(\mathfrak{sl}_2)$ a central element of the form $q^{(K+2)D}S$, deforming the difference $(K+2)D - L_{-1}$, where K and D are the central and derivation elements of an extension of the loop algebra, L_{-1} is a mode of the Sugawara tensor, and S belongs to the subalgebra of $DY(\mathfrak{sl}_2)$ "without D". The construction of this element follows from a general construction of V. Drinfeld in [4] of central elements in quasi-triangular Hopf algebras, implementing isomorphisms of modules with their double duals.

We then show that S plays a role similar to L_{-1} in the classical situation, with infinitesimal shifts replaced by finite shifts of the points: its copy $S^{(i)}$ on the ith

factor of $DY(\mathfrak{sl}_2)^{\otimes n}$ conjugates the subalgebra $U_\hbar\mathfrak{g}_{\mathbf{z}}$ to $U_\hbar\mathfrak{g}_{\mathbf{z}+\hbar(K+2)\delta_i}$, where δ_i in the ith basis vector of $\mathbb{C}^n$ (see sect. 3.3.3).

As in the classical case, the actions of the $S^{(i)}$ define a discrete flat connection on the space of coinvariants $H_0(U_\hbar\mathfrak{g}_{\mathbf{z}}, \mathbb{V})^*$, where $\mathbb{V}$ is the representation induced to $DY(\mathfrak{sl}_2)^{\otimes n}$ from a finite-dimensional representation of $U_\hbar\mathfrak{g}_{\mathcal{O}}$. We then compute explicitly this connection (sect. 3.3), and find that it agrees with the quantum KZ connection of [11].

1. Algebras associated with $\mathfrak{sl}_2$ in the rational case

1.1. The algebra $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$.

1.1.1. *Manin triples.* Let us fix an integer $n \geq 1$. Let z_i, $i = 1, \dots, n$ be a family of complex numbers; set $\mathbf{z} = (z_i)$. Let t be the standard coordinate on $\mathbb{C}P^1$, and set $t_i = t - z_i$; t_i is then a local coordinate at z_i.

Let $\mathcal{K}_i = \mathbb{C}((t_i))$ and $\mathcal{O}_i = \mathbb{C}[[t_i]]$ be the local field and ring at z_i. Let us set

$$\mathcal{K} = \oplus_{i=1}^n \mathcal{K}_i, \quad \mathcal{O} = \oplus_{i=1}^n \mathcal{O}_i,$$

and let $R_{\mathbf{z}}$ be the subring of $\mathcal{K}$, formed by the expansions of regular functions on $\mathbb{C}P^1 - \{z_i\}$, vanishing at infinity. A basis of $R_{\mathbf{z}}$ is formed by the $(t - z_i)^{-k-1}$, $i = 1, \dots, n$, $k \geq 0$; as an element of $\mathcal{K}$, $(t - z_i)^{-k-1}$ should be viewed as $((t_j + z_j - z_i)^{-k-1})_{j=1,\dots,n}$. We then have the direct sum decomposition $\mathcal{K} = R_{\mathbf{z}} \oplus \mathcal{O}$. Let us endow $\mathcal{K}$ with the scalar product $\langle \phi, \psi \rangle_{\mathcal{K}} = \sum_{i=1}^n \mathrm{res}_{z_i}(\phi\psi dz)$.

Let us set $\bar{\mathfrak{g}} = \mathfrak{sl}_2$, and construct the Lie algebras

$$\mathfrak{g}_{\mathcal{K},\mathbf{z}} = (\bar{\mathfrak{g}} \otimes \mathcal{K}) \oplus \mathbb{C}D_{\mathbf{z}} \oplus \mathbb{C}K_{\mathbf{z}}$$

as the double extension of the loop algebra $\bar{\mathfrak{g}} \otimes \mathcal{K}$ by the cocycle $c(x \otimes \phi, y \otimes \psi) = \langle x, y \rangle_{\bar{\mathfrak{g}}} \sum_{i=1}^n \mathrm{res}_{z_i}(\phi d\psi)$, and by the derivation $[D_{\mathbf{z}}, x \otimes \phi] = x \otimes (d\psi/dz)$.

This Lie algebra is endowed with the scalar product $\langle , \rangle_{\mathfrak{g}_{\mathcal{K},\mathbf{z}}}$, defined on $\bar{\mathfrak{g}} \otimes \mathcal{K}$ by

$$\langle x \otimes \phi, y \otimes \psi \rangle_{\mathfrak{g}_{\mathcal{K},\mathbf{z}}} = \langle x, y \rangle_{\bar{\mathfrak{g}}} \langle \phi, \psi \rangle_{\mathcal{K}},$$

$\langle , \rangle_{\bar{\mathfrak{g}}}$ being the Killing form of $\bar{\mathfrak{g}}$, and $\langle D_{\mathbf{z}}, K_{\mathbf{z}} \rangle_{\mathfrak{g}_{\mathcal{K},\mathbf{z}}} = 1$, $\langle D_{\mathbf{z}}, \bar{\mathfrak{g}} \otimes \mathcal{K} \rangle_{\tilde{\mathfrak{g}}_{\mathcal{K},\mathbf{z}}} = \langle K_{\mathbf{z}}, \bar{\mathfrak{g}} \otimes \mathcal{K} \rangle_{\tilde{\mathfrak{g}}_{\mathcal{K},\mathbf{z}}} = \langle D_{\mathbf{z}}, D_{\mathbf{z}} \rangle_{\mathfrak{g}_{\mathcal{K},\mathbf{z}}} \langle K_{\mathbf{z}}, K_{\mathbf{z}} \rangle_{\mathfrak{g}_{\mathcal{K},\mathbf{z}}} = 0$.

The Lie algebra $\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ contains subalgebras

$$\mathfrak{g}_{\mathbf{z}} = (\bar{\mathfrak{g}} \otimes R_{\mathbf{z}}) \oplus \mathbb{C}K_{\mathbf{z}}, \quad \mathfrak{g}_{\mathcal{O}} = (\bar{\mathfrak{g}} \otimes \mathcal{O}) \oplus \mathbb{C}D_{\mathbf{z}}.$$

Both subalgebras are isotropic for the scalar product $\langle \ , \ \rangle_{\mathfrak{g}_{\mathcal{K},\mathbf{z}}}$.

We then construct the Manin triple

$$\mathfrak{g}_{\mathcal{K},\mathbf{z}} = \mathfrak{g}_{\mathbf{z}} \oplus \mathfrak{g}_{\mathcal{O}}. \tag{1}$$

In [6], we also considered the following twisted Manin triples. Let $\bar{\mathfrak{g}} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a Cartan decomposition of $\bar{\mathfrak{g}}$. Set

$$\mathfrak{g}_{+,\mathbf{z}} = (\mathfrak{h} \otimes R_{\mathbf{z}}) \oplus (\mathfrak{n}_+ \otimes \mathcal{K}) \oplus \mathbb{C}K_{\mathbf{z}}, \quad \mathfrak{g}_- = (\mathfrak{h} \otimes \mathcal{O}) \oplus (\mathfrak{n}_- \otimes \mathcal{K}) \oplus \mathbb{C}D_{\mathbf{z}},$$

and let $\mathfrak{g}^{w_0}_{+,\mathbf{z}}, \mathfrak{g}^{w_0}_{\mathcal{O}}$ be the subspaces defined as $\mathfrak{g}_{+,\mathbf{z}}, \mathfrak{g}_{\mathcal{O}}$, exchanging $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Then we have the Manin triples

$$\mathfrak{g}_{\mathcal{K},\mathbf{z}} = \mathfrak{g}_{+,\mathbf{z}} \oplus \mathfrak{g}_-, \tag{2}$$

and

$$\mathfrak{g}_{\mathcal{K},\mathbf{z}} = \mathfrak{g}^{w_0}_{+,\mathbf{z}} \oplus \mathfrak{g}^{w_0}_-. \tag{3}$$

1.1.2. *Quantization of the Manin triples (2) and (3).* In [6], we defined quantizations of the Manin triples (2) and (3). Let us recall their construction.

Let $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ be the algebra generated by $D_{\mathbf{z}}, K_{\mathbf{z}}, x^{(i)}_k$, $k \in \mathbb{Z}, i = 1, \cdots, n$, $x = e, f, h$, arranged in the generating series

$$x_+(t) = \sum_{i=1}^n \sum_{k\geq 0} x^{(i)}_k (t - z_i)^{-k-1}, \quad x^{(i)}_-(t_i) = \sum_{k\geq 0} x^{[i]}_{-k-1} t_i^k,$$

where

$$x^{[i]}_{-k} = x^{(i)}_{-k} + \sum_{j\neq i, l\geq 0} (-1)^l \binom{l+k-1}{k-1} z_{ji}^{-l-k} x^{(j)}_l$$

(we note as usual $z_{ij} = z_i - z_j$); we also set

$$x^{(i)}(t_i) = \sum_{k\in\mathbb{Z}} x^{(i)}_k t_i^{-k-1};$$

we have then $\sum_{i=1}^n x^{(i)}(t_i) = x_+(z_i + t_i) + x^{(i)}_-(t_i)$; and we set

$$k_+(t) = \exp\left(\frac{q^\partial - 1}{\partial} h_+(t)\right), \quad K^{(i)}_-(t_i) = \exp\left(\hbar h^{(i)}_-(t_i)\right), \quad \partial = d/dt, \tag{4}$$

and $K_+(t) = k_+(t_i)k_+(t_i - \hbar)$, $K_-(t_i) = k^{(i)}_-(t_i)k^{(i)}_-(t_i - \hbar)$. In (4), the arguments of the exponentials are viewed as formal power series in $\hbar$, with coefficients in $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}} \bar{\otimes} R_{\mathbf{z}}$ in the first case, and in $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}} \bar{\otimes} \mathcal{O}$ in the second one. Here $\bar{\otimes}$ denotes the graded tensor product with respect to the bases (t_i^k) of $\mathcal{O}$, and $(t - z_1)^{-k_1-1} \cdots (t - z_n)^{-k_n-1}$ of $R_{\mathbf{z}}$. In the notation of [6], we have $x^{(i)}_k = x[t_i^k]$, $x^{[i]}_k = x[(t - z_i)^k]$. Thus $x^{(i)}_k$ is a q-analogue of $x \otimes t_i^k$ in $\bar{\mathfrak{g}} \otimes \mathcal{K}_i \subset \bar{\mathfrak{g}} \otimes \mathcal{K}$ and $x^{[i]}_k$ is a q-analogue of $x \otimes (t - z_i)^k \in \bar{\mathfrak{g}} \otimes R_{\mathbf{z}}$. The above relation between $x^{[i]}_k$ and the generators $x^{(j)}_l$ is obtained by the Laurent expansion at z_j.

The relations are

$$(t - z_j - u_j)k_+(t)e^{(j)}(u_j) = (t - z_j - u_j + \hbar)e^{(j)}(u_j)k_+(t), \tag{5}$$

$$(t - z_j - u_j + \hbar)k_+(t)f^{(j)}(u_j) = (t - z_j - u_j)f^{(j)}(u_j)k_+(t), \tag{6}$$

$$(z_{ij} + t_i - u_j - \hbar K_{\mathbf{z}} + \hbar)k^{(i)}_-(t_i)e^{(j)}(u_j) = (z_{ij} + t_i - u_j - \hbar K_{\mathbf{z}})e^{(j)}(u_j)k^{(i)}_-(t_i), \tag{7}$$

$$(z_{ij}+t_i-u_j)k_-^{(i)}(t_i)f^{(j)}(u_j) = (z_{ij}+t_i-u_j+\hbar)f^{(j)}(u_j)k_-^{(i)}(t_i), \tag{8}$$

$$(z_{ij}+t_i-u_j-\hbar)e^{(i)}(t_i)e^{(j)}(u_j) = (z_{ij}+t_i-u_j+\hbar)e^{(j)}(u_j)e^{(i)}(t_i), \tag{9}$$

$$(z_{ij}+t_i-u_j+\hbar)f^{(i)}(t_i)f^{(j)}(u_j) = (z_{ij}+t_i-u_j-\hbar)f^{(j)}(u_j)f^{(i)}(t_i), \tag{10}$$

$$[e^{(i)}(t_i), f^{(j)}(u_j)] = \delta_{ij}\frac{1}{\hbar}\left(\delta(t_i,u_j)K^+(z_i+t_i) - \delta(t_i, u_j-\hbar K_{\mathbf{z}})K_-^{(j)}(u_j)^{-1}\right), \tag{11}$$

$$[K_{\mathbf{z}}, \text{anything}] = [k_+(t), k_+(u)] = [K_-^{(i)}(t_i), K_-^{(j)}(t_j)] = 0, \tag{12}$$

$$\begin{aligned}(t-z_j-u_j-\hbar)&(t-z_j-u_j+\hbar K_{\mathbf{z}}+\hbar)K_+(t)K_-^{(j)}(u_j) \\ &= (t-z_j-u_j+\hbar)(t-z_j-u_j+\hbar K_{\mathbf{z}}-\hbar)K_-^{(j)}(u_j)K_+(t).\end{aligned} \tag{13}$$

$$[D_{\mathbf{z}}, x^{(i)}(t_i)] = -(dx^{(i)}/dt_i)(t_i), \quad x = e, f, h. \tag{14}$$

Here $\delta(z,w) = \sum_{k\in\mathbb{Z}} z^k w^{-k-1}$; note that we have changed both signs of $K_{\mathbf{z}}$ and $D_{\mathbf{z}}$ with respect to the convention of [5].

This algebra is endowed with Hopf structures (Δ, ε, S) and $(\bar\Delta, \varepsilon, \bar S)$, quantizing the bialgebra structures (2) and (3) respectively. These coproducts are defined by

$$\Delta(k_+(t)) = k_+(t)\otimes k_+(t), \quad \Delta(K_-^{(i)}(t_i)) = K_-^{(i)}(t_i)\otimes K_-^{(i)}(t_i-\hbar(K_{\mathbf{z}})_1), \tag{15}$$

$$\Delta(e^{(i)}(t_i)) = e^{(i)}(t_i)\otimes K_+(z_i+t_i) + 1\otimes e^{(i)}(t_i), \tag{16}$$

$$\Delta(f^{(i)}(t_i)) = f^{(i)}(t_i)\otimes 1 + K_-^{(i)}(t_i)^{-1}\otimes f^{(i)}(t_i-\hbar(K_{\mathbf{z}})_1), \tag{17}$$

$$\Delta(D_{\mathbf{z}}) = D_{\mathbf{z}}\otimes 1 + 1\otimes D_{\mathbf{z}}, \quad \Delta(K_{\mathbf{z}}) = K_{\mathbf{z}}\otimes 1 + 1\otimes K_{\mathbf{z}}, \tag{18}$$

where $(K_{\mathbf{z}})_1$ and $(K_{\mathbf{z}})_2$ mean $K_{\mathbf{z}}\otimes 1$ and $1\otimes K_{\mathbf{z}}$; and

$$\bar\Delta(k_+(t)) = k_+(t)\otimes k_+(t), \quad \bar\Delta(K_-^{(i)}(t_i)) = K_-^{(i)}(t_i)\otimes K_-^{(i)}(t_i+\hbar(K_{\mathbf{z}})_1)), \tag{19}$$

$$\bar\Delta(e^{(i)}(t_i)) = e^{(i)}(t_i-\hbar(K_{\mathbf{z}})_2)\otimes K_-^{(i)}(t_i-\hbar(K_{\mathbf{z}})_2)^{-1} + 1\otimes e^{(i)}(t_i), \tag{20}$$

$$\bar\Delta(f^{(i)}(t_i)) = f^{(i)}(t_i)\otimes 1 + K_+^{(i)}(t_i)\otimes f^{(i)}(t_i), \tag{21}$$

$$\bar\Delta(D_{\mathbf{z}}) = D_{\mathbf{z}}\otimes 1 + 1\otimes D_{\mathbf{z}}, \quad \bar\Delta(K_{\mathbf{z}}) = K_{\mathbf{z}}\otimes 1 + 1\otimes K_{\mathbf{z}}. \tag{22}$$

The counit ε is defined to be equal to zero on all generators.

Proposition 1.1. *(see [6]) The above formulas define quantizations $(U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}},\Delta)$ and $(U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}},\bar\Delta)$ of the Manin triples (2) and (3). These are quasitriangular Hopf algebras, with universal R-matrices respectively given by*

$$\mathcal{R}=q^{D_{\mathbf{z}}\otimes K_{\mathbf{z}}}\exp\left(\frac{\hbar}{2}\sum_{i=1}^n\sum_{k\geq 0}h_k^{(i)}\otimes h_{-k-1}^{[i]}\right)\exp\left(\hbar\sum_{i=1}^n\sum_{k\in\mathbb{Z}}e_k^{(i)}\otimes f_{-k-1}^{(i)}\right),$$

$$\bar{\mathcal{R}}=\exp\left(\hbar\sum_{i=1}^n\sum_{k\in\mathbb{Z}}f_k^{(i)}\otimes e_{-k-1}^{(i)}\right)q^{D_{\mathbf{z}}\otimes K_{\mathbf{z}}}\exp\left(\frac{\hbar}{2}\sum_{i=1}^n\sum_{k\geq 0}h_k^{(i)}\otimes h_{-k-1}^{[i]}\right).$$

The coproducts Δ and $\bar\Delta$ are related by the twist

$$F=\exp\left(\hbar\sum_{i=1}^n\sum_{k\in\mathbb{Z}}e_k^{(i)}\otimes f_{-k-1}^{(i)}\right);$$

this means that we have

$$\bar\Delta=\mathrm{Ad}(F)\circ\Delta.$$

We denote $\mathrm{Ad}(X)$, for an invertible element X of some algebra A, the linear endomorphism $Y\mapsto XYX^{-1}$ of A. F also satisfies the cocycle equation

$$(F\otimes 1)(\Delta\otimes 1)(F)=(1\otimes F)(1\otimes\Delta)(F). \tag{23}$$

Remark 1. The above results can be extended to the case where we take $\mathbf{z}$ in $\mathbb{C}[[\hbar]]^n$, such that $z_i-z_j\notin\hbar\mathbb{C}[[\hbar]]$ for $i\neq j$. □

1.1.3. *Subalgebras of* $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$. Let us consider in $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$, the subalgebras $U_\hbar\mathfrak{g}_{\mathcal{O}}$ and $U_\hbar\mathfrak{g}_{\mathbf{z}}$, respectively generated by $D_{\mathbf{z}}$ and the $x_k^{(i)}$, $k\geq 0$, and by $K_{\mathbf{z}}$ and the

$$x_{-k}^{[i]}=x_{-k}^{(i)}+\sum_{j\neq i,l\geq 0}(-1)^l\binom{l+k-1}{k-1}z_{ji}^{-l-k}x_l^{(j)},\quad k\geq 1;$$

$x=e,f,h$.

Proposition 1.2. *(see [7], sect. 4) The inclusion of algebras $U_\hbar\mathfrak{g}_{\mathbf{z}}\subset U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ and $U_\hbar\mathfrak{g}_{\mathcal{O}}\subset U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ are flat deformations of the inclusions of enveloping algebras $U\mathfrak{g}_{\mathbf{z}}\subset U\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ and $U\mathfrak{g}_{\mathcal{O}}\subset U\mathfrak{g}_{\mathcal{K},\mathbf{z}}$.*

Proposition 1.3. *The product maps*

$$U_\hbar\mathfrak{g}_{\mathbf{z}}\otimes U_\hbar\mathfrak{g}_{\mathcal{O}}\to U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}\ \text{ and }\ U_\hbar\mathfrak{g}_{\mathcal{O}}\otimes U_\hbar\mathfrak{g}_{\mathbf{z}}\to U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$$

define linear isomorphisms.

Proof. This follows from the fact that the three spaces $U_\hbar\mathfrak{g}_{\mathbf{z}},U_\hbar\mathfrak{g}_{\mathcal{O}}$ and $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ are flat deformations of $U\mathfrak{g}_{\mathbf{z}},U\mathfrak{g}_{\mathcal{O}}$ and $U\mathfrak{g}_{\mathcal{K},\mathbf{z}}$, and that the product maps $U\mathfrak{g}_{\mathbf{z}}\otimes U\mathfrak{g}_{\mathcal{O}}\to U\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ and $U\mathfrak{g}_{\mathcal{O}}\otimes U\mathfrak{g}_{\mathbf{z}}\to U\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ are linear isomorphisms. □

1.2. **Hopf structures on $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ and its subalgebras.** In a way similar to [5], we define the linear maps

$$\Pi_{\mathbf{z},\ell} : U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}} \to U_\hbar\mathfrak{g}_{\mathbf{z}}, \quad \Pi_{\mathcal{O},r} : U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}} \to U_\hbar\mathfrak{g}_{\mathcal{O}}$$

by posing

$$\Pi_{\mathbf{z},\ell}(a_{\mathbf{z}}a_{\mathcal{O}}) = a_{\mathbf{z}}\varepsilon(a_{\mathcal{O}}), \quad \Pi_{\mathcal{O},r}(a_{\mathbf{z}}a_{\mathcal{O}}) = \varepsilon(a_{\mathbf{z}})a_{\mathcal{O}},$$

and

$$\Pi_{\mathcal{O},\ell} : U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}} \to U_\hbar\mathfrak{g}_{\mathcal{O}}, \quad \Pi_{\mathbf{z},r} : U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}} \to U_\hbar\mathfrak{g}_{\mathbf{z}}$$

by posing

$$\Pi_{\mathcal{O},\ell}(a_{\mathcal{O}}a_{\mathbf{z}}) = a_{\mathcal{O}}\varepsilon(a_{\mathbf{z}}), \quad \Pi_{\mathbf{z},r}(a_{\mathcal{O}}a_{\mathbf{z}}) = \varepsilon(a_{\mathcal{O}})a_{\mathbf{z}},$$

for $a_{\mathbf{z}} \in U_\hbar\mathfrak{g}_{\mathbf{z}}$, $a_{\mathcal{O}} \in U_\hbar\mathfrak{g}_{\mathcal{O}}$. The fact that these maps are well-defined follows from Prop. 1.3.

We then have

Theorem 1.1. *(see [6, 8]) We have the equalities*

$$(\Pi_{\mathbf{z},r} \otimes 1)(F) = (1 \otimes \Pi_{\mathcal{O},r})(F) \text{ and } (\Pi_{\mathcal{O},\ell} \otimes 1)(F) = (1 \otimes \Pi_{\mathbf{z},\ell})(F).$$

If we set

$$F_1 = (\Pi_{\mathbf{z},r} \otimes 1)(F) = (1 \otimes \Pi_{\mathcal{O},r})(F),$$

and

$$F_2 = (\Pi_{\mathcal{O},\ell} \otimes 1)(F) = (1 \otimes \Pi_{\mathbf{z},\ell})(F),$$

we have $F = F_2F_1$. The maps $\mathrm{Ad}(F_1) \circ \Delta$ and $\mathrm{Ad}(F_2^{-1}) \circ \bar{\Delta}$ coincide; we will denote them by $\Delta_{\mathbf{z}}$. $\Delta_{\mathbf{z}}$ defines a quasitriangular Hopf algebra structure on $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$; $U_\hbar\mathfrak{g}_{\mathbf{z}}$ and $U_\hbar\mathfrak{g}_{\mathcal{O}}$ are Hopf subalgebras of $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ for this structure. The universal R-matrix of $(U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}, \Delta_{\mathbf{z}})$ is expressed by

$$\mathcal{R}_{\mathbf{z}} = F_1^{(21)} q^{D_{\mathbf{z}} \otimes K_{\mathbf{z}}} \exp\left(\frac{\hbar}{2}\sum_{i=1}^n \sum_{k\geq 0} h_k^{(i)} \otimes h_{-k-1}^{[i]}\right) F_2 \tag{24}$$

Moreover, F_1 and F_2 have the expansions

$$F_1 \in 1 + \hbar \sum_{i=1}^n \sum_{k\geq 0} e_k^{(i)} \otimes f_{-k-1}^{[i]} + U_\hbar\mathfrak{n}_+^{[2]} \otimes U_\hbar\mathfrak{n}_-^{[2]} \tag{25}$$

and

$$F_2 \in 1 + \hbar \sum_{i=1}^n \sum_{k\geq 0} e_{-k-1}^{[i]} \otimes f_k^{(i)} + U_\hbar\mathfrak{n}_+^{[2]} \otimes U_\hbar\mathfrak{n}_-^{[2]}, \tag{26}$$

where $U_\hbar\mathfrak{n}_\pm^{[2]}$ are the linear spans in $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ of products of more than two elements of the form $x[\epsilon]$, $x = e$ for $\pm = +$ and $x = f$ for $\pm = -$, $\epsilon \in \mathcal{K}$.

The Hopf algebra $(U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}, \Delta_{\mathbf{z}})$, together with its Hopf subalgebras $U_\hbar\mathfrak{g}_{\mathbf{z}}$ and $U_\hbar\mathfrak{g}_{\mathcal{O}}$, forms a quantization of the Manin triple (1).

1.3. **Representations of $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ and L-operators.** In [6], we studied level zero representations of the algebras introduced there. Our result can be expressed as follows. Define the algebra $U_\hbar\mathfrak{g}'_{\mathcal{K},\mathbf{z}}$ as the subalgebra of $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ generated by the $K_\mathbf{z}$ and the $x[\epsilon], \epsilon \in \mathcal{K}, x = e, f, h$; and $U_\hbar\mathfrak{g}_\mathbf{z}$ and $U_\hbar\mathfrak{g}'_\mathcal{O}$ as its subalgebras respectively generated by the $x[\epsilon], \epsilon \in R_\mathbf{z}$ and $K_\mathbf{z}$, and by the $x[\epsilon], \epsilon \in \mathcal{O}$, $x = e, f, h$.

Proposition 1.4. *We have an algebra morphism*

$$\pi : U_\hbar\mathfrak{g}'_{\mathcal{K},\mathbf{z}} \to \mathrm{End}(\mathbb{C}^2) \otimes \mathcal{K}[[\hbar]],$$

defined by

$$\pi(h_k^{(i)}) = \begin{pmatrix} \frac{2}{1+q^\partial} t_i^k & 0 \\ 0 & -\frac{2}{1+q^{-\partial}} t_i^k \end{pmatrix}, \quad k \geq 0,$$

$$\pi(h_k^{[i]}) = \begin{pmatrix} \left(\frac{1-q^{-\partial}}{\hbar\partial}\right)(t-z_i)^k & 0 \\ 0 & -\left(\frac{q^\partial - 1}{\hbar\partial}(t-z_i)^k\right) \end{pmatrix}, \quad k < 0,$$

$$\pi(e_k^{(i)}) = \begin{pmatrix} 0 & t_i^k \\ 0 & 0 \end{pmatrix}, \quad \pi(f_k^{(i)}) = \begin{pmatrix} 0 & 0 \\ t_i^k & 0 \end{pmatrix}, \quad k \in \mathbb{Z},$$

and $\pi(K_\mathbf{z}) = 0$, where ∂ is the derivation of $\mathcal{K}$ defined as $\sum_{i=1}^n d/dt_i$; it coincides with d/dt when restricted to $R_\mathbf{z}$.

The images by π of $U_\hbar\mathfrak{g}_\mathbf{z}$ and $U_\hbar\mathfrak{g}'_\mathcal{O}$ are contained in $\mathrm{Id}_{\mathbb{C}^2} \otimes \mathbb{C}[[\hbar]] + \hbar\,\mathrm{End}(\mathbb{C}^2) \otimes R_\mathbf{z}[[\hbar]]$ and $\mathrm{End}(\mathbb{C}^2) \otimes \mathcal{O}[[\hbar]]$ respectively.

Define the L-operators of $(U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}, \Delta_\mathbf{z})$ as

$$L_\mathbf{z}^+ = (\pi \otimes 1)(\mathcal{R}_\mathbf{z}^{(21)}), \quad L_\mathbf{z}^- = (\pi \otimes 1)(\mathcal{R}_\mathbf{z}^{-1} q^{D_\mathbf{z} \otimes K_\mathbf{z}}).$$

Since $\mathcal{R}_\mathbf{z}$ is contained in the completion of $U_\hbar\mathfrak{g}_\mathcal{O} \otimes U_\hbar\mathfrak{g}_\mathbf{z}$, see (24), and has leading term 1, we have

$$L_\mathbf{z}^+ \in 1 + \hbar(\mathrm{End}(\mathbb{C}^2) \otimes R_\mathbf{z})\bar{\otimes} U_\hbar\mathfrak{g}_\mathcal{O}, \quad L_\mathbf{z}^- \in 1 + \hbar(\mathrm{End}(\mathbb{C}^2) \otimes \mathcal{O})\bar{\otimes} U_\hbar\mathfrak{g}_\mathbf{z}.$$

In what follows, we will stress the functional dependences of $L_\mathbf{z}^+$ and $L_\mathbf{z}^-$ by writing $L_\mathbf{z}^+$ as $L_\mathbf{z}^+(t)$ and the component of $L_\mathbf{z}^-$ in $\mathcal{O}_i$ as $L_{(i)}^-(t_i)$.

From (25) and (26) follows that $L_\mathbf{z}^+(t)$ and $L_{(i)}^-(t_i)$ are decomposed as follows:

$$L_\mathbf{z}^+(t) = \begin{pmatrix} 1 & \hbar f_+(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_+(t-\hbar) & 0 \\ 0 & k_+^{-1}(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \hbar e_+(t) & 1 \end{pmatrix}, \tag{27}$$

and

$$L_{(i)}^-(t_i) = \begin{pmatrix} 1 & -\hbar f_-^{(i)}(t_i) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_-^{(i)}(t_i-\hbar)^{-1} & 0 \\ 0 & k_-^{(i)}(t_i) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\hbar e_-^{(i)}(t_i+\hbar K_\mathbf{z}) & 1 \end{pmatrix}. \tag{28}$$

1.4. RLL **relations for** $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ **and its subalgebras.** Let us compute the image by $\pi\otimes\pi$ of the universal R-matrix of $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$.

We find

$$(\pi\otimes\pi)(\mathcal{R}_{\mathbf{z}})(t_i,u)=R^-(z_i+t_i,u), \tag{29}$$

where

$$R^-(z_i+t_i,u)=\exp\left(\sum_{k\geq 0}(\frac{1}{\partial}\frac{q^\partial-1}{q^\partial+1}t_i^k)(u-z_i)^{-k-1}\right)\cdot \tag{30}$$
$$\cdot\frac{1}{z_i+t_i-u-\hbar}((z_i+t_i-u)\operatorname{Id}_{\mathbb{C}^2\otimes\mathbb{C}^2}-\hbar P),$$

expanded at the vicinity of $t_i=0$, where P is the permutation operator of the two factors of $\mathbb{C}^2\otimes\mathbb{C}^2$.

By applying the representation π to two out of the three factors of the Yang-Baxter equation

$$\mathcal{R}_{\mathbf{z}}^{(12)}\mathcal{R}_{\mathbf{z}}^{(13)}\mathcal{R}_{\mathbf{z}}^{(23)}=\mathcal{R}_{\mathbf{z}}^{(23)}\mathcal{R}_{\mathbf{z}}^{(13)}\mathcal{R}_{\mathbf{z}}^{(12)},$$

we obtain the relations

$$R(t,t')L_{\mathbf{z}}^{+(1)}(t)L_{\mathbf{z}}^{+(2)}(t')=L_{\mathbf{z}}^{+(2)}(t')L_{\mathbf{z}}^{+(1)}(t)R(t,t'), \tag{31}$$

$$R(z_i+t_i,t')L_{(i)}^{-(1)}(t_i)L_{\mathbf{z}}^{+(2)}(t')=L_{\mathbf{z}}^{+(2)}(t')L_{(i)}^{-(1)}(t_i)R(z_i+t_i,t'+\hbar K_{\mathbf{z}}), \tag{32}$$

$$R(z_{ij}+t_i,t_j)L_{(i)}^{-(1)}(t_i)L_{(j)}^{-(2)}(t_j)=L_{(j)}^{-(2)}(t_j)L_{(i)}^{-(1)}(t_i)R(z_{ij}+t_i,t_j), \tag{33}$$

with

$$R(z,z')=\frac{1}{z-z'+\hbar}((z-z')\operatorname{Id}_{\mathbb{C}^2\otimes\mathbb{C}^2}+\hbar P)\exp\left(-\sum_{k\geq 0}\left((\frac{1}{\partial}\frac{q^\partial-1}{q^\partial+1})z^k\right)z'^{-k-1}\right); \tag{34}$$

we set

$$R(z)=\frac{1}{z+\hbar}(z\operatorname{Id}_{\mathbb{C}^2\otimes\mathbb{C}^2}+\hbar P)\exp\left((\frac{1}{\partial}\frac{q^\partial-1}{q^\partial+1})z^{-1}\right),$$

so that $R(z,z')=R(z-z')$.

Moreover, we have the relations

$$\det_\hbar(L_{\mathbf{z}}^+(t))=\det_\hbar(L_{(i)}^-(t_i))=1, \tag{35}$$

where

$$\det_\hbar(L(t))=l_{11}(t)l_{22}(t-\hbar)-l_{12}(t)l_{21}(t-\hbar),$$

for $L(t)$ a matrix with entries $l_{ij}(t)$; this follows from the identities

$$k^+(t)e^+(u)k^+(t)^{-1}=\frac{t-u+\hbar}{t-u}e^+(u)-\frac{\hbar}{t-u}e^+(t),$$

$$k^-(t_i)^{-1}e^-(u_i+\hbar K_{\mathbf{z}})k^-(t_i)^{-1} = \frac{t_i-u_i+\hbar}{t_i-u_i}e^-(u_i-\hbar K_{\mathbf{z}}) - \frac{\hbar}{t_i-u_i}e^-(t_i),$$

specialized to $t=u-\hbar$ and $t_i=u_i-\hbar$.

Below we will denote by 1_i the element of $\mathcal{K}$ with ith component equal to 1 and the others equal to 0.

Proposition 1.5. *The algebra with generators $D_{\mathbf{z}}, K_{\mathbf{z}}$ and $l^{(i)}_{\alpha\beta}[k]$, $\alpha,\beta=1,2,k\in\mathbb{Z}, i=1,\dots,n$, arranged in matrices*

$$L^+_{\alpha\beta}(t) = \delta_{\alpha\beta}+\hbar\sum_{i=1}^{n}\sum_{k\geq 0} l^{(i)}_{\alpha\beta}[k](t-z_i)^{-k-1}, \quad L^{-(i)}_{\alpha\beta}(t_i) = \delta_{\alpha\beta}1_i+\hbar\sum_{k\geq 0} l^{(i)}_{\alpha\beta}[-k-1]t_i^k,$$

subject to relations (31), (32), (33), (35), and

$$[K_{\mathbf{z}},\text{anything}]=0, \quad [D_{\mathbf{z}},L^+_{\alpha\beta}(t)] = -\frac{dL^+_{\alpha\beta}(t)}{dt}, \quad [D_{\mathbf{z}},L^{-(i)}_{\alpha\beta}(t_i)] = -\frac{dL^{-(i)}_{\alpha\beta}(t_i)}{dt_i}, \tag{36}$$

is isomorphic to $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$.

Proof. Denote by U the algebra defined above. As we have seen before, the formulas (27) and (28) define a morphism from U to $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$. On the other hand, because of relations (35), a system of generators of U is given by $D_{\mathbf{z}}, K_{\mathbf{z}}$ and the $l^{(i)}_{\alpha\beta}[k]$, $(\alpha,\beta)\neq(2,2)$, $i=1,\dots,n,k\in\mathbb{Z}$. Relations (31), (32) and (33) allow to write commutators $[l^{(i)}_{\alpha\beta}[k], l^{(i')}_{\alpha'\beta'}[k']]$ as linear combinations of the $l^{(i)}_{\alpha\beta}[k]$, up to first order in $\hbar$; it is easy to check that this combination is given by the Lie algebra bracket in $\bar{\mathfrak{g}}\otimes\mathcal{K}$. Therefore, U is a deformation of the tensor product of the symmetric algebra of $\bar{\mathfrak{g}}\otimes\mathcal{K}$ with the symmetric algebra in $D_{\mathbf{z}}$ and $K_{\mathbf{z}}$. In the classical limit, the morphism defined by (27) and (28) is the identity. □

Corollary 1.1. *Let φ be an algebra morphism from $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ to some algebra $\mathcal{A}$. Then the images by $\varphi\otimes 1\otimes 1$ of $L^+(t)$ and $L^-_{(i)}(t_i)$ are elements*

$$\varphi(L^+(t)) \in 1\otimes 1+\hbar(\mathcal{A}\otimes\mathrm{End}(\mathbb{C}^2))\bar{\otimes}R_{\mathbf{z}}[[\hbar]], \tag{37}$$

$$\varphi(L^-_{(i)}(t_i)) \in 1\otimes 1_i+\hbar(\mathcal{A}\otimes\mathrm{End}(\mathbb{C}^2))\bar{\otimes}\mathcal{O}[[\hbar]], \tag{38}$$

satisfying (32), (32), (35). Conversely, any matrices (37), (38) satisfying these equations define a morphism $\varphi: U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}\to\mathcal{A}$.

1.5. **Isomorphism of $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ with $DY(\mathfrak{sl}_2)^{\otimes n}/(K^{(i)}-K^{(j)})$.** In this section, we will construct an isomorphism $i_{\mathbf{z}}$ from $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ to $DY(\mathfrak{sl}_2)^{\otimes n}/(K^{(i)}-K^{(j)})$. We set $K^{(i)}=1^{\otimes(i-1)}\otimes K\otimes 1^{\otimes(n-i)}$ and we will denote by K the common value of the $K^{(i)}$ in the latter algebra.

1.5.1. *Presentation of* $DY(\mathfrak{sl}_2)$. We will express $i_{\mathbf{z}}$ in terms of L-operators. To do that, let us remark that there is a simple presentation of $DY(\mathfrak{sl}_2)$ as $U_\hbar\mathfrak{g}_{\mathbb{C}((t)),(0)}$, the specialization for $n=1$ and $\mathbf{z}=(0)$ of the algebra $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$.

In terms of L-operators, $DY(\mathfrak{sl}_2)$ is presented as follows. It has generators D, K and $l_{\alpha\beta}[k]$, $\alpha,\beta=1,2,k\in\mathbb{Z}$, generating series

$$L^+_{\alpha\beta}(z)=\delta_{\alpha\beta}+\hbar\sum_{k\geq 0}l_{\alpha\beta}[k]z^{-k-1},\quad L^-_{\alpha\beta}(z)=\delta_{\alpha\beta}+\hbar\sum_{k\geq 0}l_{\alpha\beta}[-k-1]z^{k},$$

and relations

$$R(z,z')L^{\pm(1)}(z)L^{\pm(2)}(z')=L^{\pm(2)}(z')L^{\pm(1)}(z)R(z,z'), \tag{39}$$

$$R(z,z')L^{-(1)}(z)L^{+(2)}(z')=L^{+(2)}(z')L^{-(1)}(z)R(z,z'-\hbar K), \tag{40}$$

where $R(z,z')$ is defined by (34), the quantum determinant relations

$$\det_\hbar(L^\pm(z))=1, \tag{41}$$

and

$$[K,\text{anything}]=0,\quad [D,L^\pm(z)]=-dL^\pm(z)/dz.$$

(In the notation of [5], $L^-(z)$ and $L^+(z)$ correspond respectively to $L^{<0}(z)$ and to the inverse of $L^{\geq 0}(z)$ in $\mathrm{End}(\mathbb{C}^2)[[z^{\pm 1}]]\bar{\otimes}DY(\mathfrak{sl}_2)$.)

We also have the relation

$$R(z,z')L^{+(1)}(z)L^{-(2)}(z')=L^{-(2)}(z')L^{+(1)}(z)R(z,z'+\hbar K).$$

1.5.2. *The isomorphism.* Let $L_i^\pm(z)$ be the image in

$$DY(\mathfrak{sl}_2)^{\otimes n}\otimes\mathrm{End}(\mathbb{C}^2)[[z^{\pm 1}]][[\hbar]]$$

of the operator $L^\pm(z)$, which belongs to $DY(\mathfrak{sl}_2)\otimes\mathrm{End}(\mathbb{C}^2)[[z^{\pm 1}]][[\hbar]]$, by the map $\ell\otimes a\mapsto 1^{\otimes(i-1)}\otimes\ell\otimes 1^{\otimes(n-i)}\otimes a$.

Consider the expression

$$L_1^+(t_i+z_{i1}-\hbar K)\cdots L_{i-1}^+(t_i+z_{i,i-1}-\hbar K)L_i^-(t_i)L_{i+1}^+(t_i+z_{i,i+1})\cdots L_n^+(t_i+z_{i,n}).$$

$L_i^-(t_i)$ belongs to $1\otimes 1_i+\hbar DY(\mathfrak{sl}_2)^{\otimes n}[[t_i]]$. On the other hand, the series $L_j^+(t_i+z_{ij}), j\neq i$ are expanded as sums $1\otimes 1+\hbar\sum_{k\geq 1}(t_i+z_{ij})^{-k}\times$ coefficients, and we expand them in turn at the vicinity of $t_i=0$. The coefficient of each power of t_i in this expansion is then an infinite series, converging in the topology of $DY(\mathfrak{sl}_2)^{\otimes n}$ because it involves large Fourier indices. Therefore the above expression belongs to $1\otimes 1_i+\hbar DY(\mathfrak{sl}_2)^{\otimes n}[[t_i]]$.

Consider now the expression

$$L_1^+(t-z_1)\cdots L_n^+(t-z_n).$$

Since each $L^+(t)$ belongs to $1\otimes 1+\hbar t^{-1}DY(\mathfrak{sl}_2)[[t^{-1}]]$, this product belongs to $1\otimes 1+\hbar DY(\mathfrak{sl}_2)^{\otimes n}\bar{\otimes}R_{\mathbf{z}}$.

Then

Proposition 1.6. *The formulas*

$$i_{\mathbf{z}}(L^-_{(i)}(t_i)) = L_1^+(t_i + z_{i1}) \cdots L^+_{i-1}(t_i + z_{i,i-1}) L_i^-(t_i) L^+_{i+1}(t_i + z_{i,i+1} - \hbar K) \cdots \\ L_n^+(t_i + z_{i,n} - \hbar K), \tag{42}$$

$$i_{\mathbf{z}}(L^+_{\mathbf{z}}(t)) = L_1^+(t - z_1) \cdots L_n^+(t - z_n), \tag{43}$$

$$i_{\mathbf{z}}(D_{\mathbf{z}}) = \sum_{i=1}^n D^{(i)}, \quad i_{\mathbf{z}}(K_{\mathbf{z}}) = K, \tag{44}$$

define an isomorphism between $U_\hbar \mathfrak{g}_{\mathcal{K},\mathbf{z}}$ and $DY(\mathfrak{sl}_2)^{\otimes n}/(K^{(i)} - K^{(j)})$.

Proof. By the above remarks, the right sides of (42) and (43) have the form prescribed by Cor. 1.1. We then check that the right sides of formulas (42) and (43) satisfy relations (31), (32) and (33). This follows from the Yangian relations (39) and (40).

The fact that the right sides of (42) and (43) satisfy the identities (35) follows from the fact that the quantum determinant is a group-like element of the Yangian algebra $Y(\mathfrak{gl}_2)$.

The relations (36) are obviously satisfied by the right sides of (44), (42) and (43). □

1.6. **Shifts of the points.** By Prop. 1.6, we have subalgebras $i_{\mathbf{z}}(U_\hbar \mathfrak{g}_{\mathbf{z}})$ of

$$DY(\mathfrak{sl}_2)^{\otimes n}/(K^{(i)} - K^{(j)}),$$

for $\mathbf{z} \in \mathbb{C}[[\hbar]]^n$, such that $z_i - z_j \notin \hbar\mathbb{C}[[\hbar]]$ for $i \neq j$.

Proposition 1.7. *Let $D^{(i)}$ be the element of $DY(\mathfrak{sl}_2)^{\otimes n}$ equal to $1^{\otimes(i-1)} \otimes D \otimes 1^{\otimes(n-i)}$. We have*

$$\mathrm{Ad}(q^{\lambda D^{(i)}})(i_{\mathbf{z}}(U_\hbar \mathfrak{g}_{\mathbf{z}})) = i_{\mathbf{z}+\hbar\lambda\delta_i}(U_\hbar \mathfrak{g}_{\mathbf{z}+\lambda\hbar\delta_i}),$$

for $\lambda \in \mathbb{C}$ and δ_i the ith standard basis vector of $\mathbb{C}^n$.

Proof. $i_{\mathbf{z}}(U_\hbar \mathfrak{g}_{\mathbf{z}})$ is generated by the coefficients of (42). We have for any $j = 1, \dots, n$,

$$\mathrm{Ad}(q^{\lambda D^{(i)}})(i_{\mathbf{z}}(L^-_{(j)}(t_j))) = i_{\mathbf{z}+\hbar\lambda\delta_i}(L^-_{(j)}(t_j - \hbar\lambda\delta_{ij}));$$

these are generating functions for $i_{\mathbf{z}+\hbar\lambda\delta_i}(U_\hbar \mathfrak{g}_{\mathbf{z}+\hbar\lambda\delta_i})$. □

2. Quantum deformation of the zero-mode of the Sugawara field

2.1. Quantum Casimir elements. In [4], V. Drinfeld proved the following fact:

Proposition 2.1. *(see [4], Prop. 2.2) Let $\mathcal{A}$ be a quasitriangular Hopf algebra with coproduct $\Delta_{\mathcal{A}}$, counit $\varepsilon_{\mathcal{A}}$ and antipode $S_{\mathcal{A}}$. Let $\mathcal{R}_{\mathcal{A}}$ be its R-matrix and set $\mathcal{R}_{\mathcal{A}}^{-1} = \sum_i c_i \otimes d_i$. Let*

$$u = \sum_i d_i S_{\mathcal{A}}(c_i).$$

Then we have for any x in $\mathcal{A}$, $S_{\mathcal{A}}^{-2}(x) = uxu^{-1}$.

In particular, if for some other $u_0 \in \mathcal{A}$, we have $S_{\mathcal{A}}^{-2}(x) = u_0 x u_0^{-1}$, then $u_0^{-1}u$ belongs to the center of $\mathcal{A}$.

2.2. Application to the deformation of the zero-mode of the Sugawara tensor. Let $\bar{\mathfrak{g}} = \mathfrak{sl}_2$ and

$$\mathfrak{g} = (\bar{\mathfrak{g}} \otimes \mathbb{C}((t))) \oplus \mathbb{C}D \oplus \mathbb{C}K$$

be the double extension of the loop algebra $\bar{\mathfrak{g}} \otimes \mathbb{C}((t))$ by the cocycle

$$c(x \otimes \phi, y \otimes \psi) = \langle x, y \rangle_{\bar{\mathfrak{g}}} \operatorname{res}_0(d\phi \psi) K,$$

($\langle , \rangle_{\bar{\mathfrak{g}}}$ is the Killing form of $\bar{\mathfrak{g}}$), and by the derivation $[D, x \otimes \phi] = x \otimes (d\phi/dt)$.

Let $\widetilde{U\mathfrak{g}}$ be the completion of the enveloping algebra of $\mathfrak{g}$, defined by the left ideals generated by the $\bar{\mathfrak{g}} \otimes t^N \mathbb{C}[[t]]$.

Let e, f, h be the Chevalley basis of $\mathfrak{sl}_2$, and let us set $x_n = x \otimes t^n$ for $x \in \mathfrak{sl}_2$. Then it is a known fact that

$$(K+2)D + \sum_{k \geq 0} e_{-k-1} f_k + f_{-k-1} e_k + \frac{1}{2} h_{-k-1} h_k$$

belongs to the center of $\widetilde{U\mathfrak{g}}$. The sum in this expression is the zero-mode of the Sugawara tensor.

Let $DY(\mathfrak{sl}_2)$ be the double Yangian algebra associated with $\mathfrak{sl}_2$. As we have seen, this algebra is generated by central and derivation elements K and D, and elements $\widetilde{x}$ lifting elements x of $\bar{\mathfrak{g}} \otimes \mathbb{C}((t))$. Let $DY(\mathfrak{sl}_2)'$ be the subalgebra of $DY(\mathfrak{sl}_2)$ generated by K and the $\widetilde{x}$. The universal R-matrix of $DY(\mathfrak{sl}_2)$ is expressed as

$$\mathcal{R}_{\mathrm{Yg}} = q^{D \otimes K} \mathcal{R}^0_{\mathrm{Yg}}, \quad \mathcal{R}^0_{\mathrm{Yg}} \in (DY(\mathfrak{sl}_2)')^{\bar{\otimes} 2}$$

(we set $q = e^{\hbar}$). Moreover, $\mathcal{R}^0_{\mathrm{Yg}}$ has the expansion

$$\mathcal{R}^0_{\mathrm{Yg}} = 1 + \hbar \sum_{k \geq 0} \widetilde{e}_k \otimes \widetilde{f}_{-k-1} + \widetilde{f}_k \otimes \widetilde{e}_{-k-1} + \frac{1}{2} \widetilde{h}_k \otimes \widetilde{h}_{-k-1} + O(\hbar^2).$$

The antipode S_{Yg} of $DY(\mathfrak{sl}_2)$ satisfies $S^2_{\mathrm{Yg}} = \mathrm{Ad}(q^{2D})$.

Let us set

$$(\mathcal{R}^0_{\mathrm{Yg}})^{-1} = \sum_i c_i^0 \otimes d_i^0,$$

$c_i^0, d_i^0 \in DY(\mathfrak{sl}_2)'$.

Proposition 2.2. *The sum*

$$T = \sum_i q^{2D} d_i^0 q^{KD} S_{\mathrm{Yg}}(c_i^0)$$

is a central element of $DY(\mathfrak{sl}_2)$. It is written as

$$T = q^{(K+2)D} S = S q^{(K+2)D}, \tag{45}$$

where S belongs to the completion of the subalgebra $DY(\mathfrak{sl}_2)'$ defined by the left ideals generated by the lifts of $\bar{\mathfrak{g}} \otimes t^N \mathbb{C}[[t]]$. Its expansion in powers of $\hbar$ is

$$T = 1 + \hbar \left((K+2)D + \sum_{k\geq 0} \widetilde{e}_{-k-1}\widetilde{f}_k + \widetilde{f}_{-k-1}\widetilde{e}_k + \frac{1}{2}\widetilde{h}_{-k-1}\widetilde{h}_k \right) + O(\hbar^2). \tag{46}$$

Proof. The first statement follows from Prop. 2.1. We then have $S = \sum_i (q^{-KD} d_i^0 q^{KD}) S_{\mathrm{Yg}}(c_i^0)$. Since $\mathcal{R}^0_{\mathrm{Yg}}$ commutes with $D \otimes 1 + 1 \otimes D$, S commutes with D. This proves (45). (46) then follows from the above expansion of $\mathcal{R}_0$. $\square$

3. Discrete connection on coinvariants

3.1. **Induced representations.** Let $DY(\mathfrak{sl}_2)_{\geq 0}$ be the subalgebra of $DY(\mathfrak{sl}_2)$ generated by the the nonnegative Fourier generators $x_k, k \geq 0$, $x = e, f, h$. $DY(\mathfrak{sl}_2)^{\otimes n}_{\geq 0}$ is isomorphic to its image in $DY(\mathfrak{sl}_2)^{\otimes n}/(K^{(i)} - K^{(j)})$; these two algebras will be denoted the same way. Finally, we denote by $DY(\mathfrak{sl}_2)^{\otimes n}_{\geq 0}[K]$ the subalgebra of $DY(\mathfrak{sl}_2)^{\otimes n}/(K^{(i)} - K^{(j)})$ generated by $DY(\mathfrak{sl}_2)^{\otimes n}_{\geq 0}$ and K. We then have:

Lemma 3.1. *$i_{\mathbf{z}}$ restricts to an isomorphism between $U_\hbar \mathfrak{g}_{\mathcal{O}}$ and $DY(\mathfrak{sl}_2)^{\otimes n}_{\geq 0}[K]$.*

Proof. It follows from (43) and (36) that $i_{\mathbf{z}}(U_\hbar \mathfrak{g}_{\mathcal{O}})$ is contained in $DY(\mathfrak{sl}_2)^{\otimes n}_{\geq 0}[K]$. Since the classical limits of both algebras coincide with $U\mathfrak{g}_{\mathcal{O}}$ and the classical limit of $i_{\mathbf{z}}$ is then the identity, $i_{\mathbf{z}}$ is an isomorphism between these algebras. $\square$

The restriction to $DY(\mathfrak{sl}_2)_{\geq 0}$ of the Yangian version of π can be specialized to $t = 0$. Denote by (V, ρ_V) the resulting 2-dimensional representation.

Lemma 3.2. *We have*

$$(id \otimes \rho_V)(L^+(t)) = R(t).$$

Proof. We have

$$(id \otimes \rho_V)(L^+(t)) = (\pi_t \otimes \rho_V)(\mathcal{R}_{\mathrm{Yg}}^{(21)}) = (R^-(-t))^{(21)} = R(t),$$

where the third equality follows from (29). $\square$

We will consider also the dual representation ρ_{V^*} defined on V^* by $\rho_{V^*}(x) = \rho_V(S_{\mathrm{Yg}}^{-1}(x))^t$, where t denotes the transposition.

We have then

Lemma 3.3.

$$(id \otimes \rho_{V^*})(L^+(t)) = (R(t)^{t_2})^{-1},$$

where the exponent t_2 denotes the transposition in the second factor.

Proof. We have (see [4], Prop. 3.1) $(S_{\mathrm{Yg}} \otimes 1)(\mathcal{R}_{\mathrm{Yg}}) = \mathcal{R}_{\mathrm{Yg}}^{-1}$, so that $(1 \otimes \rho_V)(\mathrm{Id} \otimes S_{\mathrm{Yg}})(L^+(t)) = R(t)^{-1}$. On the other hand, $(\mathrm{Id} \otimes S_{\mathrm{Yg}}^{-2})(L^+(t)) = L^+(t + 2\hbar)$. Therefore, $(\mathrm{Id} \otimes \rho_V)(\mathrm{Id} \otimes S_{\mathrm{Yg}}^{-1})(L^+(t)) = R(t + 2\hbar)^{-1}$. So we have

$$(\mathrm{Id} \otimes \rho_V^t)(\mathrm{Id} \otimes S_{\mathrm{Yg}}^{-1})(L^+(t)) = (R(t + 2\hbar)^{-1})^{t_2}.$$

The result now follows from the identity

$$(R(t + 2\hbar)^{-1})^{t_2} = (R(t)^{t_2})^{-1}.$$

$\square$

Let us fix now a complex number k and consider the module $(\rho, (V^*)^{\otimes n})$ over $DY(\mathfrak{sl}_2)_{\geq 0}^{\otimes n}[K]$, defined as follows. As an algebra, $DY(\mathfrak{sl}_2)_{\geq 0}^{\otimes n}[K]$ is isomorphic to $DY(\mathfrak{sl}_2)_{\geq 0}^{\otimes n} \otimes \mathbb{C}[K]$. $DY(\mathfrak{sl}_2)_{\geq 0}^{\otimes n}$ then acts on $(V^*)^{\otimes n}$ by $\rho_{V^*}^{\otimes n}$, and K acts on this space by the scalar k.

Let $\mathbb{V}$ be the induced module

$$\mathbb{V} = (DY(\mathfrak{sl}_2)')^{\otimes n}/(K^{(i)} - K^{(j)}) \otimes_{DY(\mathfrak{sl}_2)_{\geq 0}^{\otimes n}[K]} (V^*)^{\otimes n};$$

this is a module over the algebra $(DY(\mathfrak{sl}_2)')^{\otimes n}/(K^{(i)} - K^{(j)})$.

3.2. **Coinvariants.** We will be interested in the (dual to the) space of coinvariants

$$H_0(U_\hbar \mathfrak{g}_{\mathbf{z}}, \mathbb{V})^*,$$

which is defined as the subspace of $\mathbb{V}^*$ consisting of the forms $\ell_{\mathbf{z}}$, such that $\ell_{\mathbf{z}}(i_{\mathbf{z}}(x)v) = \varepsilon(x)\ell_{\mathbf{z}}(v)$, for $x \in U_\hbar \mathfrak{g}_{\mathbf{z}}$.

Props. 1.3 and 1.6 show that the map from $(V^*)^{\otimes n}$ to $\mathbb{V}$, sending v to $1 \otimes v$, is injective. This map will serve to identify $(V^*)^{\otimes n}$ with a subspace of $\mathbb{V}$.

Proposition 3.1. *The restriction of $\ell_{\mathbf{z}}$ to $(V^*)^{\otimes n}$ defines a map from $H_0(U_\hbar \mathfrak{g}_{\mathbf{z}}, \mathbb{V})^*$ to $((V^*)^{\otimes n})^* = V^{\otimes n}$. This map is a linear isomorphism.*

Proof. It follows from Props. 1.6 and 1.3 that a basis of $\mathbb{V}$ is given by the $u_i \otimes v_j$, with $(u_i), (v_j)$ bases of $U_\hbar \mathfrak{g}_{\mathbf{z}}$ and $(V^*)^{\otimes n}$ respectively $(u_0 = 1)$. A form $\ell_{\mathbf{z}}$ of $\mathbb{V}^*$ is then invariant iff it satisfies the equations $\ell_{\mathbf{z}}(u_i \otimes v_j) = \varepsilon(u_i)\ell_{\mathbf{z}}(v_j)$, so it is exactly determined by the $\ell_{\mathbf{z}}(1 \otimes v_j)$. $\square$

3.3. **Compatible difference system on coinvariants.**

3.3.1. *Definitions.* Let $\mathbb{C}^n_*$ be the complement of the diagonals in $\mathbb{C}^n$, and let η be a nonzero complex number. A difference flat connection on $\mathbb{C}^n_*$ is the data of a vector space $E_{\mathbf{z}}$ for each $\mathbf{z} \in \mathbb{C}^n_*$, together with a system of linear isomorphisms

$$A_i(\mathbf{z}) : E_{\mathbf{z}} \to E_{\mathbf{z}+\eta\delta_i},$$

satisfying the relations

$$A_j(\mathbf{z}+\eta\delta_i) \circ A_i(\mathbf{z}) = A_i(\mathbf{z}+\eta\delta_j) \circ A_j(\mathbf{z}), \quad \forall i, j = 1, \dots, n; \tag{47}$$

η is called the step of the system.

Suppose we have a system $\iota_{\mathbf{z}}$ of isomorphisms of the $E_{\mathbf{z}}$ with a fixed vector space E. We get a system of elements $\widetilde{A}_i(\mathbf{z}) = \iota_{\mathbf{z}+\hbar\delta_i} A_i(\mathbf{z}) \iota_{\mathbf{z}}^{-1} \in \mathrm{Aut}(E)$, satisfying the same relations. Such a system is called a compatible difference system (see [1, 13]).

3.3.2. *The case of a formal step.* We will need the following modification of the above definitions in the formal context. In that situation, $\mathbf{z}$ is a sequence $(z_1, \dots, z_n) \in \mathbb{C}[[\hbar]]^n$, such that $z_i \neq z_j$ mod $\hbar$ for $i \neq j$. The $E_{\mathbf{z}}$ are free $\mathbb{C}[[\hbar]]$-modules and η belongs to $\hbar\mathbb{C}[[\hbar]]$.

A compatible difference system will be a system of $\mathbb{C}[[\hbar]]$-linear isomorphisms $A_i(\mathbf{z}) : E_{\mathbf{z}} \to E_{\mathbf{z}+\eta\delta_i}$ satisfying (47); a system of $\mathbb{C}[[\hbar]]$-linear isomorphisms $\iota_{\mathbf{z}} : E_{\mathbf{z}} \to E$ then gives rise to a system of elements $\widetilde{A}_i \in \mathrm{Aut}_{\mathbb{C}[[\hbar]]}(E)$, satisfying the same relations, again called a compatible difference system. (In the case where E has rank p, $\mathrm{Aut}_{\mathbb{C}[[\hbar]]}(E)$ is isomorphic to $GL_p(\mathbb{C}[[\hbar]])$.)

3.3.3. *The action of the quantum Sugawara element on coinvariants.* We then have a compatible difference system, in the formal sense, defined as follows. Associate to $\mathbf{z}$ the coinvariants space $E_{\mathbf{z}} = H_0(U_\hbar\mathfrak{g}_{\mathbf{z}}, \mathbb{V})^*$. Set $\eta = \hbar(k+2)$ and define

$$A_i(\mathbf{z}) : E_{\mathbf{z}} \to E_{\mathbf{z}+\hbar(k+2)\delta_i}$$

by the formula

$$(A_i(\mathbf{z})\ell_{\mathbf{z}})(v) = \ell_{\mathbf{z}}(\rho(S^{(i)})v), \tag{48}$$

for $\ell_{\mathbf{z}} \in E_{\mathbf{z}}, v \in \mathbb{V}_\lambda$, where $1^{\otimes(i-1)} \otimes S \otimes 1^{\otimes(n-i)}$ and S is the quantum Sugawara element defined in Prop. 2.2.

Let us show that the form $A_i(\mathbf{z})\ell_{\mathbf{z}}$ in invariant with respect to $U_\hbar\mathfrak{g}'_{\mathcal{K},\mathbf{z}+\hbar(k+2)\delta_i}$. For $x \in U_\hbar\mathfrak{g}'_{\mathcal{K},\mathbf{z}+\hbar(k+2)\delta_i}$, we have

$$\begin{aligned}(A_i(\mathbf{z})\ell_{\mathbf{z}})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x)v) &= \ell_{\mathbf{z}}(S^{(i)} i_{\mathbf{z}+\hbar(k+2)\delta_i}(x)v) \\ &= \ell_{\mathbf{z}}(\mathrm{Ad}(S^{(i)})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x))S^{(i)}v);\end{aligned}$$

but since $q^{(K+2)D}S$ is central in $DY(\mathfrak{sl}_2)$, we have $\mathrm{Ad}(S^{(i)})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x)) = \mathrm{Ad}(q^{-(K+2)D^{(i)}})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x))$; the action of this element on $\mathbb{V}$ coincides with that of $\mathrm{Ad}(q^{-(k+2)D^{(i)}})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x))$, which belongs to $i_{\mathbf{z}}(U_\hbar\mathfrak{g}_{\mathbf{z}})$ by Prop. 1.7.

We then have

$$\begin{aligned}(A_i(\mathbf{z})\ell_{\mathbf{z}})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x)v) &= \ell_{\mathbf{z}}(\mathrm{Ad}(q^{-(k+2)D^{(i)}})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x))v) \\ &= \varepsilon(\mathrm{Ad}(q^{-(k+2)D^{(i)}})(i_{\mathbf{z}+\hbar(k+2)\delta_i}(x)))\ell_{\mathbf{z}}(v) \\ &= \varepsilon(x)\ell_{\mathbf{z}}(v).\end{aligned}$$

This shows that $A_i(\mathbf{z})\ell_{\mathbf{z}}$ is $U_\hbar\mathfrak{g}_{\mathbf{z}+\hbar(k+2)\delta_i}$-invariant.

4. Identification with the qKZ system

The aim of this section is to make the system (48) explicit, using the identifications of Prop. 3.1 of the spaces of coinvariants with $V^{\otimes n}$.

4.1. **Expression of the Sugawara action in terms of L-operators.**

Let us express the connection (48) in terms of L-operators. For that, we will prove:

Lemma 4.1. *Let us write $L^-(\hbar k) = \sum_\alpha a_\alpha \otimes l_\alpha^-$, with $a_\alpha \in \mathrm{End}(V)[[\hbar]]$ and $l_\alpha^- \in DY(\mathfrak{sl}_2)$. Let v belong to $(V^*)^{\otimes n}$; we view it as a vector of $\mathbb{V}$, as explained above. Then the action of $S^{(i)}$ on v is expressed as*

$$\rho(S^{(i)})v = \sum_\alpha \rho((l_\alpha^-)^{(i)})(a_\alpha^t)^{(i)}v;$$

recall that the exponent i means the action on the ith factor of $(V^*)^{\otimes n}$.

Proof. In the notation of sect. 2.2, we have $S = \sum_i (q^{-KD}d_i^0 q^{KD})S_{\mathrm{Yg}}(c_i^0)$. Therefore,

$$\rho(S^{(i)}) = \sum_j \rho(q^{-kD}d_j^0 q^{kD})(1^{\otimes(i-1)} \otimes \rho_{V^*}(S_{\mathrm{Yg}}(c_j^0)) \otimes 1^{\otimes(n-i)}). \tag{49}$$

On the other hand, we have $(1\otimes q^{-kD})L^-(t)(1\otimes q^{kD}) = L^-(t+\hbar k)$, so that for $t=0$ we obtain, since ρ_V coincides with the specialization of π for $t=0$,

$$\sum_i \rho_V(c_j^0) \otimes q^{-KD}d_j^0 q^{KD} = \sum_\alpha a_\alpha \otimes l_\alpha^-.$$

Therefore, we have

$$\sum_i \rho_{V^*}(S_{\mathrm{Yg}}(c_j^0)) \otimes q^{-KD}d_j^0 q^{KD} = \sum_\alpha a_\alpha^t \otimes l_\alpha^-.$$

The lemma follows from the comparison of this formula with (49). ∎

4.2. **Expression of the discrete compatible system.** Let us express the invariance of the form $\ell_{\mathbf{z}}$. Let V^a be an auxiliary copy of the vector space V. Consider $i_{\mathbf{z}}(L^-_{(i)}(t_i))$ as an elements of $\mathrm{End}(V^a) \otimes DY(\mathfrak{sl}_2)^{\otimes n}/(K^{(i)} - K^{(j)})$. We then have, for $v_a \in V^a, v \in (V^*)^{\otimes n}$, and any formal t_i,

$$(1 \otimes \ell_{\mathbf{z}}) \Big((1 \otimes \rho)(i_{\mathbf{z}}(L^-_{(i)}(t_i)))(v_a \otimes v) \Big) = (1 \otimes \ell_{\mathbf{z}})(v_a \otimes v).$$

Substitute t_i by $\hbar k$ in this identity. Using (42), we find that

$$\begin{aligned}\sum_\alpha (1 \otimes \ell_{\mathbf{z}}) \big(&(R^{(a1)}(\hbar k + z_{i,1})^{t_1})^{-1} \cdots (R^{(a,i-1)}(\hbar k + z_{i,i-1})^{t_{i-1}})^{-1} \rho(l^{-(i)}_\alpha) a^{(a)}_\alpha \\ &(R^{(a,i+1)}(z_{i,i+1})^{t_{i+1}})^{-1} \cdots (R^{(an)}(z_{i,n})^{t_n})^{-1} (v^a \otimes v) \big) \\ &= (1 \otimes \ell_{\mathbf{z}})(v^a \otimes v).\end{aligned}$$

The exponent t_i denotes here the transposition of the ith factor.

Therefore, we have the identity

$$\begin{aligned}&\sum_\alpha (1 \otimes \ell_{\mathbf{z}}) \big((R^{(a1)}(\hbar k + z_{i,1})^{t_1})^{-1} \cdots (R^{(a,i-1)}(\hbar k + z_{i,i-1})^{t_{i-1}})^{-1} \rho(l^{-(i)}_\alpha) a^{(a)}_\alpha \\ &(v^a \otimes v)) \\ &= (1 \otimes \ell_{\mathbf{z}}) \big(R^{(a,n)}(z_{i,n})^{t_n} \cdots R^{(a,i+1)}(z_{i,i+1})^{t_{i+1}} (v^a \otimes v) \big) .\end{aligned} \tag{50}$$

for any $v^a \in V^a$, $v \in (V^*)^{\otimes n}$.

Introduce now

$$\tilde{R}(z) = \Big(\big(R(z)^{-1}\big)^{t_2} \Big)^{-1}$$

This object has the following property: if we write $(R(z)^{t_2})^{-1} = \sum_i x'_i \otimes x''_i$ and $\tilde{R}(z) = \sum_i y'_i \otimes y''_i$, then $\sum_{ij} y'_i x'_j \otimes x''_j y''_i = \mathrm{Id}_{V \otimes V^*}$. Thus we have

$$\begin{aligned}&\sum_\alpha (1 \otimes \ell_{\mathbf{z}}) \big(\rho(l^{-(i)}_\alpha) a^{(a)}_\alpha (v^a \otimes v) \big) \\ &= (1 \otimes \ell_{\mathbf{z}}) \big(\tilde{R}^{(a,i-1)}(\hbar k + z_{i,i-1}) \cdots \tilde{R}^{(a1)}(\hbar k + z_{i,1}) \\ &R^{(a,n)}(z_{i,n})^{t_n} \cdots R^{(a,i+1)}(z_{i,i+1})^{t_{i+1}} (v^a \otimes v) \big).\end{aligned} \tag{51}$$

Note that we have $\tilde{R}(z) = R(z + 2\hbar)^{t_2}$, so that

$$\begin{aligned}&\sum_\alpha (1 \otimes \ell_{\mathbf{z}}) \big(\rho(l^{-(i)}_\alpha) a^{(a)}_\alpha (v^a \otimes v) \big) \\ &= (1 \otimes \ell_{\mathbf{z}}) \big(R^{(a,i-1)}(z_{i,i-1} + \hbar(k+2))^{t_{i-1}} \cdots R^{(a,1)}(z_{i,1} + \hbar(k+2))^{t_1} \\ &R^{(a,n)}(z_{i,n})^{t_n} \cdots R^{(a,i+1)}(z_{i,i+1})^{t_{i+1}} (v^a \otimes v) \ \big) .\end{aligned}$$

After we apply the transposition on V^a, we obtain, for any $u^a \in V^*$,

$$\sum_\alpha (1 \otimes \ell_{\mathbf{z}}) \left(\rho(l_\alpha^{-(i)})(a_\alpha^{(a)})^t(u^a \otimes v)\right) \tag{52}$$
$$= (1 \otimes \ell_{\mathbf{z}}) \left(R^{(a,i+1)}(z_{i,i+1})^t \cdots R^{(a,n)}(z_{i,n})^t \right.$$
$$\left. R^{(a,1)}(z_{i,1} + \hbar(k+2))^t \cdots R^{(a,i-1)}(z_{i,i-1} + \hbar(k+2))^t(u^a \otimes v)\ \right).$$

Here $R(z)^t = R(z)^{t_1 t_2}$ is the transposed of $R(z)$.

Let $v_1, \dots, v_n$ belong to V^*. Let (e_β) be a basis of V^* and (e^β) be the dual basis. Let us apply the identity (52) to

$$u^a \otimes v = v_i \otimes (v_1 \otimes \cdots \otimes v_{i-1} \otimes e_\beta \otimes v_{i+1} \otimes \cdots \otimes v_n),$$

apply $e^\beta \otimes 1$ to this identity, and sum over β. We find, with $v_{1,i-1} = v_1 \otimes \cdots \otimes v_{i-1}$ and $v_{i+1,n} = v_{i+1} \otimes \cdots \otimes v_n$,

$$\sum_\alpha \sum_\beta (e^\beta \otimes \ell_{\mathbf{z}}) \left(\rho(l_\alpha^{-(i)})(a_\alpha^{(a)})^t(v_i \otimes v_{1,i-1} \otimes e_\beta \otimes v_{i+1,n})\right) \tag{53}$$
$$= \sum_\beta (e^\beta \otimes \ell_{\mathbf{z}}) \left(R^{(a,i+1)}(z_{i,i+1}) \cdots R^{(a,n)}(z_{i,n}) \right.$$
$$\left. R^{(a,1)}(z_{i,1} + \hbar(k+2)) \cdots R^{(a,i-1)}(z_{i,i-1} + \hbar(k+2))(v_i \otimes v_{1,i-1} \otimes e_\beta \otimes v_{i+1,n})\ \right). \tag{54}$$

The following result is a consequence of the definition of dual bases.

Lemma 4.2. *For any endomorphism X of V^*, and any element v of V^*, we have*

$$\sum_\alpha (e^\beta \otimes 1)(Xv \otimes e_\beta) = Xv.$$

Applying this lemma to (52), we find

$$\sum_\alpha \ell_{\mathbf{z}}(\rho(l_\alpha^{-(i)})(a_\alpha^{(i)})^t v) = \ell_{\mathbf{z}} \left(R^{(i,i+1)}(z_{i,i+1})^t \cdots R^{(in)}(z_{i,n})^t \right. \tag{55}$$
$$\left. R^{(i1)}(z_{i,1} + \hbar(k+2))^t \cdots R^{(i,i-1)}(z_{i,i-1} + \hbar(k+2))^t v\right).$$

By Lemma 4.1, the left side of this equality is the expression of $(A_i \ell_{\mathbf{z}})(v)$.

Theorem 4.1. *The discrete flat connection on the spaces of coinvariants defined by (48) is identified by the isomorphisms (3.1) with the quantum Knizhnik-Zamolodchikov system*

$$A_i \ell_{\mathbf{z}} = R^{(i,i-1)}(z_{i,i-1} + \hbar(k+2)) \cdots R^{(i1)}(z_{i,1} + \hbar(k+2)) \tag{56}$$
$$R^{(in)}(z_{i,n}) \cdots R^{(i,i+1)}(z_{i,i+1}) \ell_{\mathbf{z}}.$$

with step $\eta = \hbar(k+2)$.

Remark 2. We have obtained equations for the coinvariants of the representation $\mathbb{V}$ of $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ induced from $\rho_{V^*}^{\otimes n}$. It should be possible to obtain equations for coinvariants of a representation $\otimes_{i=1}^n \rho_{V_i^*}$, where V_i are finite dimensional representations of $DY(\mathfrak{sl}_2)_{\geq 0}$. For that, one should consider the L-operators $L_{(i)}^{-(V_i)}(t_i)$ for $U_\hbar\mathfrak{g}_{\mathcal{K},\mathbf{z}}$ and $L^{\pm(V_i)}(t)$, defined by replacing π by π_{V_i} in the definitions of $L_{(i)}^{-}(t_i)$ and $L^\pm(t)$, and prove that $i_{\mathbf{z}}(L_{(i)}^{-(V_i)}(t_i))$ is given by the formula of Prop. 3.1. After that, it should be possible to apply the reasoning above to derive the general version of the qKZ equation.

Remark 3. It is interesting to consider the systems obtained by replacing $\mathbb{V}$ by a representation induced from a non-irreducible representation of $U_\hbar\mathfrak{g}_{\mathcal{O}}$. In the classical case and the $\mathfrak{gl}_1$ situation, one obtains in this way non-Fuchsian systems.

For example, $DY(\mathfrak{sl}_2)_{\geq 0}$ has an evaluation representation $\rho_{V[[\zeta]]}$ on $V[[\zeta]]$, defined by $(id \otimes \rho_{V[[\zeta]]})(L^+(t)) = R(t+\zeta)$; the representation ρ_V considered above is a quotient of $\rho_{V[[\zeta]]}$. One may expect that the system of equations one would obtain this way is the system

$$\eta = \hbar(k+2), \quad A_i \ell_{\mathbf{z}} = K_i(z_{ij} + \zeta_i - \zeta_j)\ell_{\mathbf{z}},$$

where $\ell_{\mathbf{z}}$ belongs to $V^{\otimes n}[[\zeta_1, \dots, \zeta_n]]$, on which the operators $K_i(z_{ij} + \zeta_i - \zeta_j)$ act as $\sum_{k\geq 0} K_i^{(k)}(z_{ij})(\zeta_i - \zeta_j)^k/k!$ (the functions of ζ_i act by multiplication on the formal series part and the derivatives of K_i act as matrices on the factor $V^{\otimes n}$; K_i are the operators appearing in the right side of (56)).

Remark 4. In the classical case, it is possible to explain the agreement of the intertwiners and coinvariants approaches in a simple way. It would be interesting to find such an explanation of the result of this paper.

References

[1] K. Aomoto, A note on holonomic q-difference systems, Algebraic analysis I, M. Kashiwara, T. Kawai, eds., Academic Press, New-York, 25-8 (1988).
[2] D. Bernard, On the Wess-Zumino-Witten model on Riemann surfaces, Nucl. Phys. B 309 (1988),145-74.
[3] V.G. Drinfeld, Quantum groups, Proc. ICM Berkeley, vol. 1, AMS, 1986, 789-820.
[4] V.G. Drinfeld, On almost cocommutative Hopf algebras, Leningrad Math. Jour. 1:2 (1990), 321-42.
[5] B. Enriquez, G. Felder, A construction of Hopf algebra cocycles for the double Yangian $DY(\mathfrak{sl}_2)$, q-alg/9703012.
[6] B. Enriquez, V. Rubtsov, Quantum groups in higher genus and Drinfeld's new realizations method, q-alg/9601022, to appear in Ann. Sci. Ec. Norm. Sup.
[7] B. Enriquez, V. Rubtsov, Quasi-Hopf algebras associated with $\mathfrak{sl}_2$ and curves of genus > 1, q-alg/9608005.
[8] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras III, q-alg/9610030.

[9] G. Felder, Conformal field theory and integrable systems associated to elliptic curves, Proc. ICM-94, 1247-55, Birkhäuser (1995).

[10] G. Felder, V. Tarasov, A. Varchenko, Monodromy of solutions of elliptic qKZB equations, q-alg/9705017.

[11] I. Frenkel, N. Reshetikhin, Quantum affine algebras and holonomic difference equations, Comm. Math. Phys. 146 (1992), 1-60.

[12] V.G. Knizhnik, A.B. Zamolodchikov, Current algebras and Wess-Zumino models in two dimensions, Nucl. Phys. B247, 83-103 (1984).

[13] V. Tarasov, A. Varchenko, Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras, Inv. Math. (1997), to appear.

[14] A. Tsuchiya, Y. Kanie, Vertex operators in conformal theories on P^1 and monodromy representations of the braid group, Adv. Stud. Pure Math. 16, 297-372 (1988).

[15] A. Tsuchiya, K. Ueno, Y. Yamada, Conformal field theory on a universal family of stable curves with gauge symmetries, Adv. Stud. Pure Math. 19 (1989), 459-566.

[16] N. Reshetikhin, M. Semenov-Tian-Shansky, Central extensions of quantum current groups, Lett. Math. Phys. 19 (1990), 133-42.

Centre de Mathématiques, URA 169 du CNRS, Ecole Polytechnique, 91128 Palaiseau, France

D-Math, ETH-Zentrum, HG G46, CH-8092 Zürich, Suisse